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On the regularization procedure in classical electrodynamics

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Abstract
We consider the self-action problem in classical electrodynamics. A strict geometrical sense of commonly used renormalization of mass is made. A regularization procedure is proposed which relies on energy–momentum and angular momentum balance equations. We correct the expression for angular momentum tensor obtained by us in a previous paper (2002 *J. Phys. A: Math. Gen.* **35** 831).

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1. Introduction

In classical electrodynamics particles interact with one another through the medium of a field. The problem then becomes one of mutual determination: the field is determined by the charged particles and their motion, and the motion of the charges is determined by the field.

The principle of least action is formulated for a composite system of point-like charged particles and their electromagnetic field. It is invariant under ten infinitesimal transformations which constitute the Poincaré group. According to Noether's theorem, these symmetry properties can be used for the derivation of conservation laws, i.e. those quantities that do not change with time.

Variation on field variables gives the Maxwell equations. Liénard–Wiechert fields are the solutions of Maxwell equations with point-like sources. These 'fields' do not have degrees of freedom of their own: they are functionals of particle paths. One can substitute these *direct particle fields* [1] in the conservation laws to rewrite them in terms of particle variables. Conserved quantities place stringent requirements on the dynamics of our system. Dirac's derivation [2] of the radiation-reaction force is based upon consideration of energy–momentum conservation.

Since Maxwell energy–momentum tensor density has a singularity on a particle world line, the verification of energy–momentum conservation is not a trivial matter. The main point is a covariant splitting of the electromagnetic field's stress–energy tensor into two parts separately conserved off the world line of the particle [3]. Volume integration of the radiative part over a spacelike three-surface Σ gives the integral of the Larmor relativistic rate over particle's world line. The amount of radiated energy–momentum in Σ depends on all previous evolution of a source. While the volume integration of the bound part results the expression which depends on the 4-velocity and 4-acceleration of the source at the observation instant only (see [3, eq. (3.20)]). The bound part of electromagnetic-field momentum is permanently 'attached' to the charge and is carried along with it.

In this paper we provide a self-contained derivation of the Lorentz–Dirac equation which relies on ten conserved quantities corresponding to Poincaré invariance of a closed particles plus field system. The first attempt was made recently [4]. For the 'centre-of-mass' conserved quantity which arises from the invariance of the system under Lorentz transformations the non-covariant expression is obtained. It contradicts the Lorentz–Dirac equation. In this paper we show that the troubles are caused by the lack of splitting of the angular momentum tensor density into the bound and the radiative components. López and Villarroel [5] find the correct equation of angular momentum balance which is consistent with the Lorentz–Dirac equation. The authors decompose the angular momentum tensor density into the bound and the radiative parts separately conserved off the world line of the particle. The bound and emitted angular momenta possess similar properties to the corresponding energy–momentum quantities. Having used the López and Villarroel decomposition we correct the previous result [4] which contradicts the Lorentz–Dirac equation.

2. Preliminaries

We choose metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ for Minkowski space $\mathbb{M}_4$. We use the Heaviside–Lorentz system of units with the velocity of light $c = 1$. Summation over repeated indices is understood throughout the paper; Greek indices run from 0 to 3, and Latin indices from 1 to 3.

We suppose that the components of momentum 4-vector carried by electromagnetic field of a point-like charge are [6]

$$p_{\mathrm{em}}^{\nu}(\tau) = P\int_{\sigma(\tau)} \mathrm{d}\sigma_\mu\, T^{\mu\nu} \tag{2.1}$$

where $\mathrm{d}\sigma_\mu$ is the vectorial surface element on a hyperplane $\sigma(\tau) = \{y \in \mathbb{M}_4 : u_\mu(\tau)(y^\mu - z^\mu(\tau)) = 0\}$ which intersects a world line

$$\zeta : \mathbb{R} \to \mathbb{M}_4 \qquad s \mapsto (z^\alpha(s)) \tag{2.2}$$

at the point $(z^\alpha(\tau)) \in \zeta$. The angular momentum tensor of the electromagnetic field is written as [6]

$$M_{\mathrm{em}}^{\mu\nu}(\tau) = P\int_{\sigma(\tau)} \mathrm{d}\sigma_\alpha\, (y^\mu T^{\alpha\nu} - y^\nu T^{\alpha\mu}). \tag{2.3}$$

The components $T^{\mu\nu}$ of electromagnetic field's stress–energy tensor have a singularity on a particle trajectory. In equations (2.1) and (2.3) capital letter P denotes the principal value of the singular integral, defined by removing from $\sigma(\tau)$ an ε-sphere around the particle and then passing to the limit $\varepsilon \to 0$.

The usage of an integration hyperplane $\sigma(\tau) = \{y \in \mathbb{M}_4 : u_\mu(\tau)(y^\mu - z^\mu(\tau)) = 0\}$ which is orthogonal to the 4-velocity of the charge at the point $z^\mu(\tau)$ is very important for the manipulation of the divergent self-energy of a point charge which was called a 'renormalization of mass'. The divergent term is due to volume integration of the bound part of the electromagnetic field's stress–energy tensor [3] (the bound part of the angular momentum tensor density [5]). The regularization procedure involves the Taylor expansion of the bound integral in powers of ε in which the first two terms lead to the diverging Coulomb self-energy and the Abraham radiation reaction 4-vector, respectively. One usually adds self-energy term to the 'matter' mass of the particle and proclaims the finite structure-independent terms as those of true physical meaning. The finite terms depend on the form of the hole that is cut out from the integration hypersurface to ensure regularization. The best suited hole must be coordinate free. It is evident that the spacelike hyperplane $\sigma(\tau)$ together with the future light cone cutting out the hole [6, Fig. 5-2], [3, Fig. 1], [5, Fig. 1] is coordinate free.

We make a Lorentz transformation such that a tilted hyperplane $\sigma(\tau)$ becomes $\Sigma_{t'} = \{y \in \mathbb{M}_4 : y^{0'} = t'\}$. The Lorentz matrix, $\Lambda(\tau)$, determines the transformation to the particle's momentarily co-moving Lorentz frame (MCLF) where the particle is momentarily at rest at observation instant τ. On rearrangement, energy–momentum (2.1) and angular momentum (2.3) take the form

$$p^{\nu}_{\text{em}}(\tau) = \Lambda^{\nu}{}_{\nu'}(\tau) P \int_{\Sigma_{t'}} \mathrm{d}\sigma_{0'}\, T^{0'\nu'} \tag{2.4}$$

$$M^{\mu\nu}_{\text{em}}(\tau) = \Lambda^{\mu}{}_{\mu'}(\tau)\Lambda^{\nu}{}_{\nu'}(\tau) P \int_{\Sigma_{t'}} \mathrm{d}\sigma_{0'}\, (y^{\mu'} T^{0'\nu'} - y^{\nu'} T^{0'\mu'}). \tag{2.5}$$

3. Energy–momentum of the retarded Liénard–Wiechert field

We see that the volume integration over hyperplane $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ is intimately connected with the integration over coordinate-free hyperplane $\sigma(\tau) = \{y \in \mathbb{M}_4 : u_\mu(\tau)(y^\mu - z^\mu(\tau)) = 0\}$. In [4] the amount of electromagnetic-field momentum of the retarded Liénard–Wiechert field in hyperplane Σ_t at fixed instant t is calculated. Volume integration of the radiative part over Σ_t gives the integral of the Larmor relativistic rate over particle's world line [4, eqs. (2.20)]. The volume integration of the bound part results the expressions which depend on the state of the particle's motion in the vicinity of the instant of observation (see [4, eqs. (2.21), (2.22)]):

$$p^0_{\text{bnd}} = \frac{2}{3}e^2 \lim_{u\to t}\left(-\frac{1}{4} + \frac{1}{1-v^2(u)}\right)\frac{1}{t-u} \qquad p^i_{\text{bnd}} = \frac{2}{3}e^2 \lim_{u\to t}\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u}. \tag{3.1}$$

To provide the regularization we expand these (divergent) expressions in powers of $\varepsilon = t-u$ and neglect all positive powers of this small parameter:

$$\begin{aligned} P^0_{\text{bnd}} &= \frac{2}{3}e^2 \lim_{\varepsilon\to 0}\left(-\frac{1}{4} + \frac{1}{1-v^2(t)}\right)\frac{1}{\varepsilon} - \frac{4}{3}e^2 a^0(t) \\ P^i_{\text{bnd}} &= \frac{2}{3}e^2 \lim_{\varepsilon\to 0}\frac{v^i(t)}{1-v^2(t)}\frac{1}{\varepsilon} - \frac{2}{3}e^2[a^i(t) + v^i(t)a^0(t)]. \end{aligned} \tag{3.2}$$

Here

$$a^0(t) = \frac{(\mathbf{v}\dot{\mathbf{v}})}{(1-v^2(t))^2} \qquad a^i(t) = \frac{\dot{v}^i(t)}{1-v^2(t)} + \frac{(\mathbf{v}\dot{\mathbf{v}})v^i(t)}{(1-v^2(t))^2} \tag{3.3}$$

are the components of particle's 4-acceleration in the 'laboratory' frame.

The choice of coordinate-dependent hole in Σ_t around the particle results the expressions with non-covariant structureless terms. What should be done about? Taking $v^i(t) = 0$ we arrive at

$$P^{0'}_{\text{bnd}} = \frac{e^2}{2\varepsilon} \qquad P^{i'}_{\text{bnd}} = -\frac{2}{3}e^2 a^{i'} \tag{3.4}$$

where the components $a^{i'} = \Lambda^{i'}{}_{\alpha} a^{\alpha}$ constitute 3-vector $\mathbf{a}$ which is (non-trivial) spatial part of the particle acceleration taken in MCLF. After that we perform the Lorentz transformation:

$$p^{\mu}_{\rm bnd} = \Lambda^{\mu}{}_{\mu'} P^{\mu'}_{\rm bnd} = \frac{e^2}{2\varepsilon} u^{\mu} - \frac{2}{3} e^2 a^{\mu}. \tag{3.5}$$

Having done these manipulations we substitute the coordinate-free hole which is cut out in Lorentz-invariant hyperplane $\sigma(\tau)$ for the coordinate-dependent hole in the non-covariant hyperplane Σ_t (cf equations (2.1) and (2.4)).

As usual, the divergent quantity $e^2/2\varepsilon$ is linked together with the mechanical 'matter' mass of a particle, so that renormalized mass m is considered to be finite. After that we obtain Teitelboim's expression [3] for the 4-momentum $p^{\mu}_{\rm part} = p^{\mu}_{\rm mech} + p^{\mu}_{\rm bnd}$ of accelerated point-like charge:

$$p^{\mu}_{\rm part} = m u^{\mu} - \tfrac{2}{3} e^2 a^{\mu}. \tag{3.6}$$

4. Angular momentum of the retarded Liénard–Wiechert field

We now turn to the calculation of the angular momentum tensor

$$M^{\mu\nu}_{\rm em} = P \int_{\Sigma_t} {\rm d}\sigma_0 \, (y^{\mu} T^{0\nu} - y^{\nu} T^{0\mu}) \tag{4.1}$$

which is rewritten in the form of (2.5). We apply the convenient coordinate system [4, eq. (2.10)] where an integration hyperplane Σ_t becomes a surface of constant value. The retarded distance r [7] is replaced by the right-hand side of [4, eq. (2.9)] in new curvilinear coordinates. It is proportional to the difference $t - u$ of the observation time t and the retarded time u.

Following [5], we present the integrand as a sum of the radiative and the bound parts:

$$M^{\mu 0\nu}_{\rm rad} = z^{\mu}(u) T^{0\nu}_{\rm rad} - z^{\nu}(u) T^{0\mu}_{\rm rad} + (y^{\mu} - z^{\mu}(u)) T^{0\nu}_{(-3)} - (y^{\nu} - z^{\nu}(u)) T^{0\mu}_{(-3)} \tag{4.2}$$

$$M^{\mu 0\nu}_{\rm bnd} = z^{\mu}(u) T^{0\nu}_{\rm bnd} - z^{\nu}(u) T^{0\mu}_{\rm bnd} + (y^{\mu} - z^{\mu}(u)) T^{0\nu}_{(-4)} - (y^{\nu} - z^{\nu}(u)) T^{0\mu}_{(-4)} \tag{4.3}$$

where $T^{0\mu}_{(-3)}$ and $T^{0\mu}_{(-4)}$ denote the parts of $T^{0\mu}_{\rm bnd}$ which are scaled as r^{-3} and r^{-4}, respectively. The components $T^{0\mu}$ of energy–momentum tensor density in terms of new coordinates are written in [4, eqs. (2.15), (2.16)].

The integration reveals that the decomposition is meaningful. Indeed, the contribution of radiative angular momentum density is regular:

$$\begin{aligned} M^{ij}_{\rm rad} = \frac{2}{3} e^2 \int_{-\infty}^{t} {\rm d}u \, \mathbf{a}^2(u) [z^i(u) v^j(u) - z^j(u) v^i(u)] \\ + \frac{2}{3} e^2 \int_{-\infty}^{t} {\rm d}u [v^i(u) a^j(u) - v^j(u) a^i(u)] \end{aligned} \tag{4.4}$$

$$M^{0i}_{\rm rad} = \frac{2}{3} e^2 \int_{-\infty}^{t} {\rm d}u \, \mathbf{a}^2(u) [u v^i(u) - z^i(u)] + \frac{2}{3} e^2 \int_{-\infty}^{t} {\rm d}u [a^i(u) - v^i(u) a^0(u)]. \tag{4.5}$$

The amount of emitted angular momentum in hyperplane Σ_t at fixed observation instant t depends on all previous motion of a source. The radiation part detaches itself from the charge and leads an independent existence, while the bound part of the angular momentum depends on the state of the source in the vicinity of the instant of observation:

$$M^{ij}_{\rm bnd} = \frac{2}{3} e^2 \lim_{u \to t} \left[z^i(u) \frac{v^j(u)}{1 - v^2(u)} \frac{1}{t - u} - z^j(u) \frac{v^i(u)}{1 - v^2(u)} \frac{1}{t - u} \right] \tag{4.6}$$

$$M^{0i}_{\rm bnd} = \frac{2}{3}e^2 \lim_{u\to t}\left[u\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u} - z^i(u)\left(-\frac{1}{4}+\frac{1}{1-v^2(u)}\right)\frac{1}{t-u}\right]. \tag{4.7}$$

The expression for mixed spacetime components $M^{0i}_{\rm em} = M^{0i}_{\rm bnd} + M^{0i}_{\rm rad}$ of angular momentum, obtained in [4, eq. (3.3)], is different from the sum of expressions (4.5) and (4.7) of the present paper. So far as radiative terms are concerned, the difference is the integral being a function of the end points only:

$$\frac{2}{3}e^2\int_{-\infty}^{t}{\rm d}u\left[\frac{\dot v^i(u)}{1-v^2(u)} + 2\frac{(\mathbf{v}\dot{\mathbf{v}})v^i(u)}{(1-v^2(u))^2}\right] = \frac{2}{3}e^2\int_{-\infty}^{t}{\rm d}u\frac{\rm d}{{\rm d}u}\left[\frac{v^i(u)}{1-v^2(u)}\right]. \tag{4.8}$$

It compensates the difference of the bound parts:

$$\begin{aligned}
&\frac{2}{3}e^2\left[\lim_{u\to t} u\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u} - t\lim_{u\to t}\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u}\right]\\
&\qquad = \frac{2}{3}e^2\int_{-\infty}^{t}{\rm d}u\left[\frac{\rm d}{{\rm d}u}\left(u\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u}\right) - t\frac{\rm d}{{\rm d}u}\left(\frac{v^i(u)}{1-v^2(u)}\frac{1}{t-u}\right)\right]\\
&\qquad = -\frac{2}{3}e^2\int_{-\infty}^{t}{\rm d}u\frac{\rm d}{{\rm d}u}\left[\frac{v^i(u)}{1-v^2(u)}\right].
\end{aligned}$$

We see that the bound and the radiative terms are mixed up in [4]. As a result the incorrect expression is obtained in this paper. It was caused by the lack of covariant splitting of angular momentum tensor density into the bound part and the radiative part [5].

We expand equations (4.6) and (4.7) in powers of $\varepsilon = t - u$. Neglecting all positive powers of ε, we arrive at the following non-covariant expressions:

$$\begin{aligned}
M^{ij}_{\rm bnd} &= z^i(t)P^j_{\rm bnd} - z^j(t)P^i_{\rm bnd}\\
M^{0i}_{\rm bnd} &= tP^i_{\rm bnd} - z^i(t)P^0_{\rm bnd} - \tfrac{2}{3}e^2\tfrac{1}{4}v^i(t).
\end{aligned} \tag{4.9}$$

Divergent components $P^\mu_{\rm bnd}$ are given by equations (3.2). To satisfy the explicit Lorentz invariance we pass to MCLF and then perform the Lorentz transformation Λ (see equation (2.5)). We obtain the covariant expression

$$M^{\mu\nu}_{\rm bnd} = z^\mu(t)p^\nu_{\rm bnd} - z^\nu(t)p^\mu_{\rm bnd} \tag{4.10}$$

where $z^\mu(t) = (t, z^i(t))$ and the components $p^\mu_{\rm bnd}$ of bound 4-momentum are given by equation (3.5).

We see that the (divergent) bound angular momentum (4.10) has precisely the same form as the mechanical angular momentum of a material particle. Therefore, the regularization of angular momentum can be reduced to the renormalization of mass.

5. Energy–momentum and angular momentum balance equations

Now we study the energy–momentum and angular momentum balance equations. To involve an external force as well as an external torque in the conservation laws we consider two sources acting on one another through the medium of the retarded Liénard–Wiechert fields. An interference of outgoing electromagnetic waves leads to the interaction between the sources (see [8] where a frontal collision of two point-like charges is examined).

There is no hyperplane which is orthogonal to the world lines of both the particles at all events. Kosyakov [9] constructs a piecewise hypersurface where a small fragment of a spacelike hyperplane Σ is replaced by a fragment of an orthogonal hyperplane $\sigma_a(\tau_a) = \left\{y \in \mathbb{M}_4 : u_{a,\mu}(\tau_a)\left(y^\mu - z^\mu_a(\tau_a)\right) = 0\right\}$ in the vicinity of every intersection point. The deformed hyperplane is called *locally adjusted*. But the problem arises how to sew these fragments with Σ.

The hyperplane $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ is convenient in this context. The 'laboratory' time t is a single common parameter defined along all the world lines of the system. The volume integration over non-deformed Σ_t of the so-called 'interference' part of Maxwell tensor density [8, eq. (2.6)] resulted in the sum of work done by Lorentz forces of point-like charges acting on one another. The integration of angular momentum tensor density [6] also leads to the sensible results [8]. Should Σ_t be deformed?

Expressions (3.1), (4.6) and (4.7) show that the volume integrals of bound parts of the electromagnetic field's stress–energy tensor and the angular momentum tensor density depend on the state of particle's motion in the vicinity of intersection point $\zeta \cap \Sigma_t$. A charged particle cannot be separated from its bound electromagnetic 'cloud' which has its own 4-momentum and angular momentum. These quantities together with their mechanical counterparts constitute the 4-momentum and angular momentum of the charged particle. We proclaim the *finite* characteristics as those of true physical meaning.

Summing up the self-action terms and the interaction terms, we arrive at the following total energy–momentum,

$$P^0 = \sum_{a=1}^{2}\left[p^0_{a,\mathrm{part}}(t) + \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\,\mathbf{a}_a^2(t_a)\right] - \sum_{b\neq a}\int_{-\infty}^{t}\mathrm{d}t_a\sqrt{1-v_a^2(t_a)}F^0_{ba} \tag{5.1}$$

$$P^i = \sum_{a=1}^{2}\left[p^i_{a,\mathrm{part}}(t) + \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\,\mathbf{a}_a^2(t_a)v_a^i(t_a)\right] - \sum_{b\neq a}\int_{-\infty}^{t}\mathrm{d}t_a\sqrt{1-v_a^2(t_a)}F^i_{ba} \tag{5.2}$$

and total angular momentum

$$\begin{aligned} M^{ij} = \sum_{a=1}^{2}\Bigg\{& z_a^i(t)p^j_{a,\mathrm{part}} - z_a^j(t)p^i_{a,\mathrm{part}} + \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\,\mathbf{a}^2(t_a)\big[z_a^i(t_a)v_a^j(t_a) - z_a^j(t_a)v_a^i(t_a)\big]\\ &+ \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\left[v_a^i(t_a)a_a^j(t_a) - v_a^j(t_a)a_a^i(t_a)\right]\Bigg\}\\ &- \sum_{b\neq a}\int_{-\infty}^{t}\mathrm{d}t_a\sqrt{1-v_a^2(t_a)}\big[z_a^i(t_a)F^j_{ba} - z_a^j(t_a)F^i_{ba}\big] \end{aligned} \tag{5.3}$$

$$\begin{aligned} M^{0i} = \sum_{a=1}^{2}\Bigg\{& tp^i_{a,\mathrm{part}} - z_a^i(t)p^0_{a,\mathrm{part}} + \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\,\mathbf{a}^2(t_a)\big[t_a v_a^i(t_a) - z_a^i(t_a)\big]\\ &+ \frac{2}{3}e_a^2\int_{-\infty}^{t}\mathrm{d}t_a\left[a_a^i(t_a) - v_a^i(t_a)a_a^0(t_a)\right]\Bigg\}\\ &- \sum_{b\neq a}\int_{-\infty}^{t}\mathrm{d}t_a\sqrt{1-v_a^2(t_a)}\left[t_a F^i_{ba} - z_a^i(t_a)F^0_{ba}\right] \end{aligned} \tag{5.4}$$

where F^μ_{ba} denotes μth component of the Lorentz force due to charge b on charge a.

The change in energy–momentum and angular momentum carried by the electromagnetic field should be balanced by a corresponding change of particles' 4-momenta and angular momenta, respectively. Since the action is not propagated instantaneously, the balance in a vicinity of the first charge as well as in a neighbourhood of the second charge should be achieved separately. The analysis of (5.1) and (5.2) gives the relativistic generalization of

Newton's second law

$$\begin{aligned}\dot{p}^0_{a,\mathrm{part}} &= -\tfrac{2}{3}e_a^2\mathbf{a}_a^2(t) + \sqrt{1-v_a^2(t)}F^0_{ba}\\ \dot{p}^i_{a,\mathrm{part}} &= -\tfrac{2}{3}e_a^2\mathbf{a}_a^2(t)v_a^i(t) + \sqrt{1-v_a^2(t)}F^i_{ba}\end{aligned} \tag{5.5}$$

where loss of energy due to radiation is taken into account. Via the differentiation of (5.3), (5.4) and taking into account the relativistic generalization of Newton's second law we arrive at the equalities which do not contain the Lorentz forces at all:

$$\begin{aligned}v_a^i(t)p^j_{a,\mathrm{part}} - v_a^j(t)p^i_{a,\mathrm{part}} &= -\tfrac{2}{3}e_a^2\left[v_a^i(t)a_a^j(t) - v_a^j(t)a_a^i(t)\right]\\ p^j_{a,\mathrm{part}} - v_a^i(t)p^0_{a,\mathrm{part}} &= -\tfrac{2}{3}e_a^2\left[a_a^i(t) - v_a^i(t)a_a^0(t)\right].\end{aligned} \tag{5.6}$$

It is convenient to rewrite these kinematic expressions in a manifestly covariant fashion:

$$u_a^\mu(\tau)p^\nu_{a,\mathrm{part}} - u_a^\nu(\tau)p^\mu_{a,\mathrm{part}} = -\tfrac{2}{3}e_a^2\left[u_a^\mu(\tau)a_a^\nu(\tau) - u_a^\nu(\tau)a_a^\mu(\tau)\right]. \tag{5.7}$$

The system of six linear equations in four variables $p^\mu_{a,\mathrm{part}}$ is equivalent to

$$p^\nu_{a,\mathrm{part}} + u_a^\nu(\tau)(u_{a,\mu}p^\mu_{a,\mathrm{part}}) = -\tfrac{2}{3}e_a^2a_a^\nu(\tau). \tag{5.8}$$

The easiest way to solve this system is the usage of MCLF where $u_a^{\nu'} = (1,0,0,0)$ and $a_a^{\nu'} = \left(0, a_a^{i'}\right)$. In MCLF $p^{\nu'}_{a,\mathrm{part}} = \left(m_a, -2\big/3e_a^2a_a^{i'}\right)$ where m_a is an arbitrary scalar. It is natural to interpret m_a as a finite rest mass of the charged particle.

In the 'laboratory' frame

$$p^\nu_{a,\mathrm{part}} = \Lambda^\nu{}_{\nu'}p^{\nu'}_{a,\mathrm{part}} = m_au_a^\nu(\tau) - \tfrac{2}{3}e_a^2a_a^\nu(\tau). \tag{5.9}$$

Therefore, Teitelboim's expression (3.6) arises from the total angular balance equations.

6. Conclusions

We construct ten conservation laws arising from the symmetry of a closed system of two point-like charged particles and their electromagnetic field under the Poincaré group. We conclude that their time derivatives completely determine the time development of the system. Indeed, energy–momentum balance equations lead to the relativistic generalization of Newton's second law where loss of energy due to radiation is taken into account, while the angular momentum balance equations are the key to the self-interaction problem. They constitute the system of six linear equations in four components of particle's momentum. Its solution is Teitelboim's expression (3.6) which contains, apart from the usual velocity term, a contribution from the acceleration when the particle is charged. Having substituted it for the particle's 4-momentum in the relativistic generalization of Newton's second law we derive the Lorentz–Dirac equation of motion of a charged particle under the influence of an external force as well as its own electromagnetic field [2].

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